

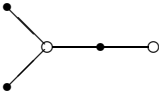
EVEN AND ODD PLANE LABELLED BIPARTITE TREES

YURY KOCHETKOV

ABSTRACT. Let $T(n, m)$ be the set of plane labelled bipartite trees with n white vertices and m — black. If the number $m + n$ of vertices is even, then the set $T(n, m)$ is a union of two disjoint subsets — subset of "even" trees and subset of "odd" trees. This partition has a clear geometric meaning.

1. INTRODUCTION

A *plane tree* is a tree embedded into plane. A *bipartite tree* is a tree with vertices colored in two colors black and white in such way that adjacent vertices have different colors. The *passport* of a bipartite tree is the non increasing sequence of degrees of its white vertices and the non increasing sequence of degrees of its black vertices.

Example 1.1. The tree  has passport $\langle 3, 1 \mid 2, 1, 1 \rangle$.

The *dual passport* of a bipartite tree is an expression of the form $(1^{i_1} 2^{i_2} \dots \mid 1^{j_1} 2^{j_2} \dots)$, where i_1, i_2, \dots are numbers of white vertices of degree 1, 2 and so on, and j_1, j_2, \dots are numbers of black vertices of degree 1, 2 and so on. The dual passport of the tree in the above example is $(1^1 3^1 \mid 1^2 2^1)$.

A labelled graph is a graph, where each vertex has a label and these labels are pairwise distinct. We will consider plane labelled bipartite trees. Here the set of labels of white vertices and the set of labels of black ones are disjoint. We will label white vertices as v_1, v_2, \dots and black as u_1, u_2, \dots .

The first problem is about enumeration: what is the number of plane labelled bipartite trees with n white vertices and m black ones?

Theorem 2.1. *The number of plane labelled bipartite trees with n white vertices and m black ones is*

$$\frac{((m + n - 2)!)^2}{(n - 1)! \cdot (m - 1)!}.$$

Remark 1.1. In what follows we will use notation (n, m) -tree to denote a bipartite trees with n white vertices and m black ones.

If the number of vertices is even, then the set of all plane labelled bipartite trees is the union of two disjoint subsets: the subset of "even" trees and the subset of "odd" ones. We correspond to a plane bipartite labelled tree T its *invariant* $i(T) = 0, 1$ (see Definition 3.1). A tree T is even, if $i(T) = 0$, and odd in the opposite case.

An *elementary movement* is a three-step procedure

- (1) we disengage a leaf, i.e. a vertex of degree one with the outgoing edge, from the adjacent vertex of the opposite color;
- (2) then we move this leaf along the tree till the next meeting of a vertex of the opposite color (it can be the same vertex, to which the leaf was attached before the move);
- (3) we attach the leaf to this vertex.

Theorem 2.1. *An elementary movement changes the parity of a tree.*

Remark 1.2. All this can be considered as a generalization of even/odd partition of the set of permutations.

2. ENUMERATION

If T is a plane (n, m) -tree, then it generates

$$\frac{n! \cdot m!}{\#\text{Aut}(T)} \quad (1)$$

labelled trees, where $\#\text{Aut}(T)$ is the order of group of automorphisms of T . Let M be a set of all plane (n, m) -trees with a fixed passport $\Pi = \langle k_1, k_2, \dots \mid l_1, l_2, \dots \rangle$ and let $\tilde{\Pi} = (1^{i_1} 2^{i_2} \dots \mid 1^{j_1} 2^{j_2} \dots)$ be the dual passport. The Goulden-Jackson theorem [1] states that

$$\sum_{T \in M} \frac{1}{\#\text{Aut}(T)} = \frac{(n-1)! \cdot (m-1)!}{i_1! \cdot \dots \cdot i_s! \cdot j_1! \cdot \dots \cdot j_s!},$$

where $s = n + m - 1$ is the number of edges.

Theorem 2.1. *The number of plane labelled (n, m) -trees is*

$$\frac{((n+m-2)!)^2}{(n-1)! \cdot (m-1)!}.$$

Proof. According to (1) and the Goulden-Jackson theorem, the number of plane labelled (n, m) -trees with the given passport $\Pi = \langle k_1, k_2, \dots \mid l_1, l_2, \dots \rangle$ is

$$n! \cdot m! \cdot \frac{(n-1)! \cdot (m-1)!}{i_1! \cdot \dots \cdot i_s! \cdot j_1! \cdot \dots \cdot j_s!}. \quad (2)$$

Numbers k_1, k_2, \dots constitute a partition of $s = n + m - 1$ of the length exactly n and numbers l_1, l_2, \dots constitute a partition of s of the length exactly m . Thus, we must sum (2) by this partitions, or sum

$$n! \cdot \frac{(n-1)!}{i_1! \cdot \dots \cdot i_s!}$$

by partitions of the length n , sum

$$m! \cdot \frac{(m-1)!}{j_1! \cdot \dots \cdot j_s!}$$

by partitions of the length m , and multiply.

Each partition $(k_1, k_2, \dots, k_n) = 1^{i_1} 2^{i_2} \dots$ of s of the length exactly n generates

$$\frac{n!}{(i_1)! \cdot (i_2)! \cdot \dots}$$

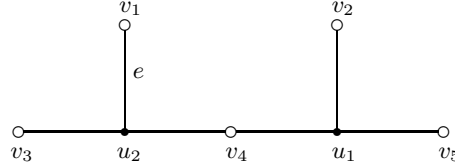
solutions of the equation $x_1 + \dots + x_n = s$, where each solution is a permutation of numbers k_1, k_2, \dots, k_n . But the number of all positive integral solutions of this equation is $\binom{s-1}{n-1}$, thus the double sum by all partitions of lengths n and m is

$$n! \cdot m! \cdot \binom{s-1}{n-1} \cdot \binom{s-1}{m-1} \cdot \frac{1}{n \cdot m} = \frac{((n+m-2)!)^2}{(n-1)! \cdot (m-1)!} \cdot \square$$

3. INVARIANT

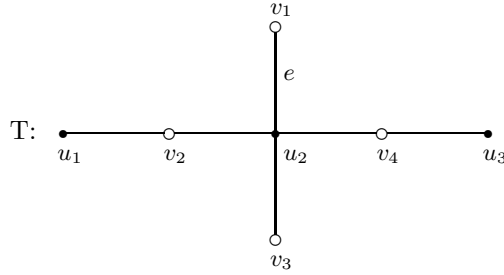
Let v_1, \dots, v_n be labels of white vertices of a plane labelled (n, m) -tree T and u_1, \dots, u_m be labels of black vertices. Some white vertex v_i will be the root vertex and some edge e , outgoing from v_i , will be the root edge. We start the counterclockwise going around of T , beginning from v_i , keeping e to the left and in the process of this going we generate the string $c(T)$ of labels and closing brackets: when we meet some vertex for the first time we write its label and when we meet it for the last time we write $)$. In $c(T)$ the number of labels is equal to the number of brackets and in each left segment the number of labels is not less than the number of brackets. Thus, in the string $c(T)$ we have a unique correspondence between labels and brackets.

Example 3.1. Let v_1 be the root vertex of the tree T



and e be its root edge. Then $c(T) = v_1 u_2 v_3) v_4 u_1 v_5) v_2))))$.

Example 3.2. Let $c(T) = v_1 u_2 v_2 u_1) v_3) v_4 u_3)))$. Then



Remark 3.1. From here we will assume that a tree has an even number of vertices.

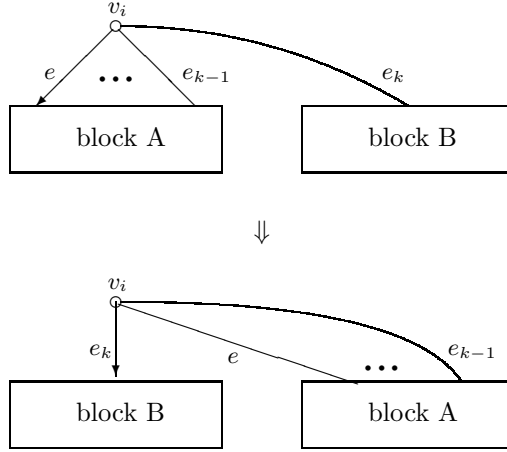
Definition 3.1. Let T be a plane labelled (n, m) -tree with the root vertex v_i and the root edge e and let $c(T)$ be the corresponding string. Also let

- a be the number of inversions in vertices v , i.e. the number of cases, when v_k is before v_l in $c(T)$, but $k > l$;
- b be the analogously defined number of inversions in vertices u ;
- c be the number of cases, when some u is before some v in $c(T)$;
- d be the number of cases, when a closing bracket is before some label;
- $e = \frac{|n-m|}{2}$.

Let $i(T) \equiv (a + b + c + \frac{d+e}{2}) \pmod 2$. A tree T will be called *even*, if $i(T) = 0$, and *odd* in the opposite case.

Theorem 3.1. *The invariant does not depend on a choice of root edge.*

Proof. Let k be the degree of root vertex v_i .



We will study the change of invariant induced by the change of a root edge, demonstrated in the figure above.

Assume that there are

- k_A "white" labels in block A and k_B "white" labels in block B, $k_A + k_B = n - 1$;
- l_A "black" labels in block A and l_B "black" labels in block B, $l_A + l_B = m$;
- x inversions in "white" labels between blocks A and B;
- y inversions in "black" labels between blocks A and B.

The change of root edge

- decreases the number of inversions in white labels by x , but increases it by $k_A k_B - x$;
- decreases the number of inversions in black labels by y , but increases it by $l_A l_B - y$;
- decreases the number of inversions in white and black labels by $l_A k_B$, but increases it by $l_B k_A$;
- does not change the number of inversions in labels and brackets.

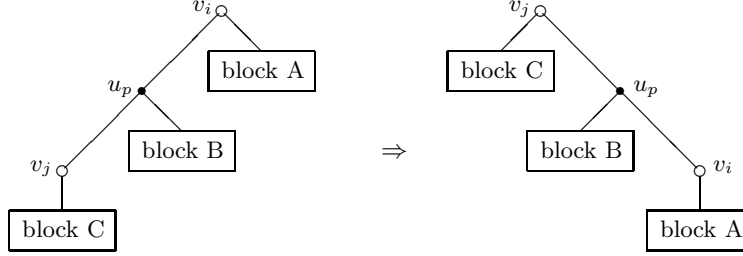
Thus, we must find the parity of the number $k_A k_B + l_A l_B - l_A k_B + l_B k_A$. Let $z = k_A + l_A$, then

$$\begin{aligned} k_A k_B + l_A l_B - l_A k_B + l_B k_A &\equiv k_A k_B + l_A l_B + l_A k_B + l_B k_A = \\ &= (k_A + l_A)(k_B + l_B) = z(n + m - 1 - z) = \\ &= (m + n)z - z(z + 1) \equiv 0 \pmod 2 \end{aligned}$$

□

Theorem 3.2. *The invariant does not depend on a choice of root vertex.*

Proof. Let the root vertex be changed from v_i to v_j :



Then the string is changed in the following way:

$$v_i u_p v_j C) B) A) \Rightarrow v_j C u_p B v_i A)))$$

As above we will assume that blocks A, B, C contain k_A "white" labels and l_A "black" labels, k_B "white" labels and l_B "black" labels, k_C "white" labels and l_C "black" labels, respectively. Then (if we do not take into account even terms) the invariant is changed by

$$\begin{aligned} & [\pm 1 + 1 + k_C + l_C + (k_C + l_C)/2 + k_B + l_B + (k_B + l_B)/2] + \\ & + [-1 - k_C + l_C + (k_C + l_C)/2] + \\ & + [-(k_A + l_A)/2 - (k_B + l_B)/2 - (k_A + l_A)/2]. \end{aligned}$$

Here terms in the first square brackets are generated by movement of v_i , in the second square brackets — by movement of u_p and in the third square brackets — by movement of two closing brackets in the string $c(T)$. Thus, the change is

$$\pm 1 - k_A - l_A + k_B + l_B + k_C + 3l_C \equiv n + m - 2 \equiv 0 \pmod{2}.$$

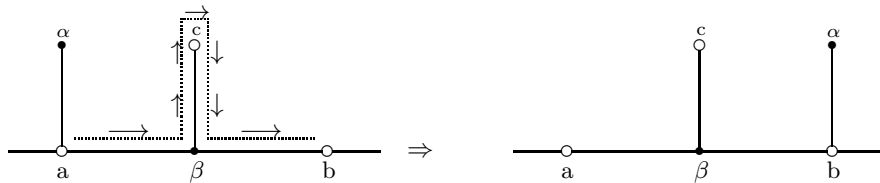
□

4. MOVEMENTS

A leaf is a vertex of degree one with the edge outgoing from it.

Definition 4.1. Let T be a plane labelled bipartite tree. A *movement* is a 3-step procedure: a) we disengage a black (white) leaf A from white (black) vertex B to which this leaf is attached; b) we move the leaf A around T clockwise or counter clockwise to a white (black) vertex C (it is possible, that C is B); c) we attach the leaf A to C . A movement of a black (white) leaf is *even*, if it bypassed an even number of black (white) vertices, and odd in the opposite case. A movement of a black (white) leaf will be called *elementary*, if it bypassed one black (white) vertex.

Example 4.1. In the figure below we see the movement of black leaf " α " from white vertex " a " to white vertex " b ".



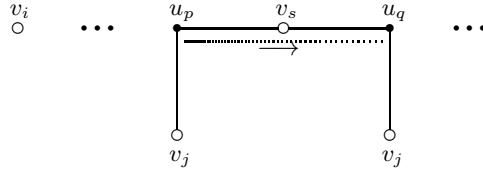
" α " bypasses black vertex " β " twice, so this movement is even.

Remark 4.1. A reason for the number of vertices to be even is that otherwise a movement of a leaf clockwise and counterclockwise to the same final position is even in one case and odd — in another.

Theorem 4.1. *An elementary movement changes the parity of a tree.*

Proof. We will check all types of elementary movements.

- A counterclockwise movement of a white leaf v_j increases the distance between the root vertex v_i and v_j by 2: at first the leaf is attached to the black vertex u_p , then it moves to the black vertex u_q , bypassing the white vertex v_s .

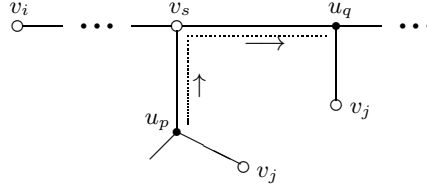


This movement changes the string in the following way:

$$\cdots v_j) v_s u_q \cdots \Rightarrow \cdots v_s u_q v_j) \cdots$$

It changes the number of inversions in white labels by 1, the number of inversions in white and black labels by 1 and the number of inversions in labels and brackets by 2, i.e. the invariant changes by $\pm 1 + 1 + 1$ — by an odd number.

- A counterclockwise movement of a white leaf v_j does not change the distance between the root vertex v_i and v_j — the movement at first decreases this distance and then increases it. In the beginning the leaf is attached to the black vertex u_p , then it moves to the black vertex u_q , bypassing the white vertex v_s .



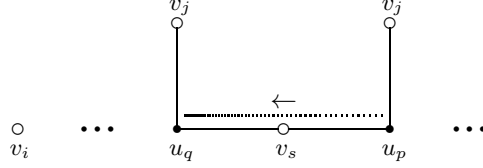
This movement changes the string in the following way:

$$\cdots v_j)) u_q \cdots \Rightarrow \cdots) u_q v_j) \cdots$$

Actually, it only changes the number of inversions in white labels and black labels by 1.

- A counterclockwise movement of a white leaf v_j does not change the distance between the root vertex v_i and v_j — the movement at first increases this distance and then decreases it. This movement in essence is an interchange of positions of two neighboring white leaves and thus only change the number of inversions in white labels by one.
- A counterclockwise movement of a white leaf v_j decreases the distance between the root vertex v_i and v_j by 2: at first the leaf is attached to the

black vertex u_p , then it moves to the black vertex u_q , bypassing the white vertex v_s .

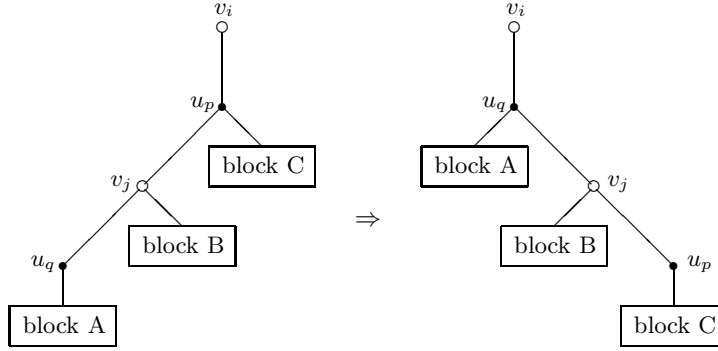


This movement changes the string in the following way:

$$\cdots v_j))) \cdots \Rightarrow \cdots))v_j) \cdots$$

It only changes the number of inversions in brackets and labels by 2, i.e. it changes the invariant by 1.

- Now let the root vertex v_i be of degree one and it makes a movement



The string is changed in the following way:

$$v_i u_p v_j u_q A) B) C)) \Rightarrow v_i u_q A v_j B u_p C)))))$$

Let us assume that

- block A contains k_A white labels (x of them precede v_j), l_A black labels (y of them precede u_p) and $k_A + l_A$ brackets;
- block B contains k_B white labels, l_B black labels (z of them precede u_p) and $k_B + l_B$ brackets;
- block C contains k_B white labels and l_B black labels.

The movement of u_p in the string changes the invariant in the following way:

- $u_p v_j \rightarrow v_j \dots u_p$: -1 ;
 - $u_p \dots u_q \rightarrow u_q \dots u_p$: ± 1 ;
 - $u_p \dots A \rightarrow A \dots u_p$: $-k_A + l_A - 2y + (k_A + l_A)/2$;
 - $u_p \dots B \rightarrow B u_p$: $-k_B + l_B - 2z + (k_B + l_B)/2$.
- $-k_A/2 + 3l_A/2 - k_B/2 + 3l_B/2 - 2y - 2z - 1 \pm 1$ in total.

The movement of v_j in the string changes the invariant in the following way:

- $v_j u_q \rightarrow u_q \dots v_j$: 1 ;
- $v_j \dots A \rightarrow A v_j$: $k_A - 2x + l_A + (k_A + l_A)/2$.

$3k_A/2 + 3l_A/2 + 1 - 2x$ in total.

Movement of brackets makes two bypasses of C and one bypass of B , thus this movement changes the invariant by $-(k_B + l_B)/2 - k_C - l_C$. Thus, the total change is

$$\begin{aligned} k_A + 3l_A - k_B + l_B - k_C - l_C - 2x - 2y - 2z \pm 1 &\equiv \\ &\equiv k_A + l_A + k_B + l_B + k_C + l_C + 1 \pmod{2}. \end{aligned}$$

It remains to note that $k_A + l_A + k_B + l_B + k_C + l_C + 1 = n + m - 3$ — odd number.

- All other cases are obvious. The analysis of the black leaf movement is the same as the analysis of white leaf movement.

□

Remark 4.2. In [2] and [3] it was proved that the set of plane bipartite weighted trees with six vertices and the given lists of white and black weights in generic case is a union of two subsets. An analytically defined invariant $I(T) = \pm\sqrt{d}$, where d is the product of weights of vertices, determines the belonging of a tree T to this or that subset. In generic case a weighted tree is a labelled tree with constraints. The study of geometrical properties of invariant $I(T)$ is the origin of this work.

REFERENCES

- [1] Goulden I.P. and Jackson D.M., The combinatorial relationship between trees, cacti and certain connection coefficients for the symmetric group, *European J. Combin.*, 1992, V.13, p. 357-365.
- [2] Kochetkov Yu., Anti-Vandermonde systems and plane trees, *Fuct. Anal. Appl.*, 36:3 (2002), 240-243.
- [3] Kochetkov Yu., Enumeration of one class of plane weighted trees, *J. Math. Sci.*, 209:2 (2015), 282-291.

E-mail address: yukochetkov@hse.ru, yuyukochetkov@gmail.com